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The Asymptotic Behaviour of Certain Interpolation Polynomials

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1. INTRODUCTION

Let f: $[-1, 1] \to R$ and let

$$x_{k,n} = \cos((2k-1)\pi/2n), \qquad k = 1, 2,..., n. \tag{1.1}$$

Now define $R_{4n-1}(f, x)$ as the unique polynomial of degree $4n - 1$, or less, such that

$$R_{4n-1}(f, x_{k,n}) = f(x_{k,n}), \qquad k = 1, 2,..., n, \tag{1.2}$$

and, for $j = 1, 2$, and 3,

$$R_{4n-1}^{(j)}(f, x_{k,n}) = 0, \qquad k = 1, 2,..., n. \tag{1.3}$$

The nodes of interpolation defined by (1.1) are the zeros of the Chebyshev polynomial

$$T_n(x) = \cos(n \arccos x), \qquad -1 \leqslant x \leqslant +1. \tag{1.4}$$

After the sequence of polynomials $\{R_{4n-1}(f, x): n = 1, 2, 3,...\}$ was first introduced by Krylov and Steuermann [3] in 1922, it was further studied in [1, 2, 4–6, 8].

In this paper we shall examine the difference $| R_{4n-1}(f, x) - f(x)|$ more closely than in [5]. In Section 2 new and old results are stated; Section 3 contains technicalities necessary for the proof of one of the new theorems and Section 4 contains the proof of that theorem.

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2. Convergence Theorems

In their study of the convergence properties of $R_{4n-1}(f, x)$, Krylov and Steuermann [3] stated the following result. (Unfortunately, their proof was incorrect: a correct proof was later given by Laden [4].)

THEOREM 1. *If $f \in C([-1, 1])$ then*

$$\lim_{n\to\infty} \|R_{4n-1}(f) - f\| = 0.$$

Here, as elsewhere in this paper, $\|\cdot\|$ denotes the uniform norm on $C([-1, 1])$.

The rate of convergence may be estimated in terms of $\omega(f; t)$, the modulus of continuity of f, which is defined by

$$\omega(f; t) := \sup\{|f(x) - f(y)| : |x - y| \leqslant t, -1 \leqslant x, y \leqslant 1\}.$$

If $\Omega(t)$ is an arbitrary modulus of continuity, then we define the class $C(\Omega)$ of functions by

$$C(\Omega) = \{f \in C([-1, 1]) : \omega(f; t) \leqslant \Omega(t) \text{ for all } t \geqslant 0\}.$$

The best result known about the rate of convergence is the following.

THEOREM 2 (T. M. Mills [5]). *There are two positive constants A and B such that, for each integer $n \geqslant 2$,*

$$\frac{A}{n} \sum_{r=2}^{n} \Omega(1/r) \leqslant \sup\{\|R_{4n-1}(f) - f\| : f \in C(\Omega)\}$$

$$\leqslant \frac{B}{n} \sum_{r=1}^{n} \Omega(1/r).$$

If we define Lip 1 $= C(\Omega_1)$, where $\Omega_1(t) \equiv t$, then Theorem 2 implies

COROLLARY 1. *If $f \in$ Lip 1, then*

$$\|R_{4n-1}(f) - f\| \leqslant C \frac{\ln n}{n} \qquad \textit{for } n = 2, 3, 4, \ldots,$$

where C is an absolute constant.

The main result of this paper is the following refinement of this corollary.

THEOREM 3. *If* $-1 \leqslant x \leqslant +1$, *then*

$$\sup\{|R_{4n-1}(f, x) - f(x)| : f \in \mathrm{Lip}\ 1\}$$
$$= \frac{4/3}{\pi} T_n(x)^4 (1 - x^2)^{1/2} \frac{\ln n}{n} + O(n^{-1}), \qquad n \to \infty. \tag{2.1}$$

Statement (2.1) highlights two facts concerning the polynomials $R_{4n-1}(f, x)$. Firstly, it is clear from (2.1) that if $f \in \mathrm{Lip}\ 1$ then

$$\begin{aligned} |R_{4n-1}(f, x) - f(x)| &= O(n^{-1}) &\qquad \text{if } |x| = 1 \\ &= O((\ln n)/n) &\qquad \text{if } x = 0; \end{aligned}$$

that is, the error is much smaller near the end points than at the centre of the interval. In this respect, it is possible to prove the more general pointwise estimate

$$|R_{4n-1}(f, x) - f(x)| \leqslant \frac{B}{n} \sum_{r=1}^{n} \Omega\left(\frac{(1 - x^2)^{1/2}}{r} + \frac{1}{r^2}\right) \tag{2.2}$$

for $-1 \leqslant x \leqslant +1$, $n \geqslant 2$ and $f \in C(\Omega)$. Here, B is an absolute constant. As the proof of (2.2) is a mere combination of the techniques in [5, 7], we shall omit it.

Secondly, the quantity $\frac{4}{3}$ in (2.1) is significant. One can prove a result similar to Theorem 3 for the well-known Hermite–Fejér interpolation polynomials based on the Chebyshev nodes (1.1). However, in this case one obtains a factor $T_n(x)^2$ (rather than the $T_n(x)^4$ in (2.1)) and the constant 2 (rather than the $\frac{4}{3}$ in (2.1)). Consequently the polynomial $R_{4n-1}(f, x)$ gives a better approximation (albeit only slightly better) than the classical Hermite–Fejér interpolation polynomial determined by the same values of the function f. Perhaps the significance of the constant $\frac{4}{3}$ is best summed up in the following corollary of Theorem 3.

COROLLARY 2.

$$\sup\{\| R_{4n-1}(f) - f \| : f \in \mathrm{Lip}\ 1\} = \frac{4}{3\pi} \cdot \frac{\ln n}{n} + O(1/n), \qquad n \to \infty.$$

3. FORMULAE

In this section we shall recall certain technical details which are necessary for the proof of Theorem 3.

The polynomial $R_{4n-1}(f, x)$ is given by

$$R_{4n-1}(f, x) = \sum_{k=1}^{n} f(x_k) S_k(x), \tag{3.1}$$

where

$$S_k(x) = F_k(x) + G_k(x) + H_k(x), \tag{3.2}$$

$$F_k(x) = \frac{1}{n^4}(1 - x_k)^2 (1 - x^2)\left(\frac{T_n(x)}{x - x_k}\right)^4, \tag{3.3}$$

$$G_k(x) = \frac{4n^2 - 1}{6n^4}(x - x_k)^2 (1 - xx_k)\left(\frac{T_n(x)}{x - x_k}\right)^4, \tag{3.4}$$

$$H_k(x) = \frac{1}{2n^4}\left(\frac{T_n(x)^2}{x - x_k}\right)^2, \tag{3.5}$$

and $x_k = x_{k,n}$ is given by (1.1).

From (1.2), (1.3), and the uniqueness of $R_{4n-1}(f, x)$ we have

$$R_{4n-1}(1, x) = \sum_{k=1}^{n} S_k(x) = 1. \tag{3.6}$$

For $-1 \leqslant x \leqslant +1$, let

$$\Delta_n(x) = \sup\{|R_{4n-1}(f, x) - f(x)| : f \in \text{Lip } 1\}$$

and

$$\phi_x(t) = |x - t|, \qquad -1 \leqslant t \leqslant +1.$$

Now, on the one hand, $\phi_x \in$ Lip 1 and hence

$$\Delta_n(x) \geqslant |R_{4n-1}(\phi_x, x) - \phi_x(x)| = \sum_{k=1}^{n} |x - x_k| S_k(x).$$

On the other hand, for any $f \in$ Lip 1, we have by (3.1) and (3.6)

$$|R_{4n-1}(f, x) - f(x)| = \left|\sum_{k=1}^{n} (f(x_k) - f(x)) S_k(x)\right| \leqslant \sum_{k=1}^{n} |x - x_k| S_k(x).$$

Therefore,

$$\Delta_n(x) = \sum_{k=1}^{n} |x - x_k| S_k(x). \tag{3.7}$$

We now estimate $\Delta_n(x)$ by a series of lemmas.

4. Estimates

In proving Theorem 3, we may assume that $x \neq x_{k,n}$, $k = 1, 2, \dots, n$, for otherwise (2.1) follows from (1.2). Define the index j by $|x - x_j| = \min\{|x - x_k| : k = 1, 2, \dots, n\}$; if there are two such values of j, then choose either one. Clearly j is a function of n and, provided $|x| \neq 1$, both j and $n - j$ are unbounded functions of n.

LEMMA 1. $\sum_{k=j-1}^{j+1} |x - x_k| S_k(x) = O(1/n)$.

If $j = 1$ or n then some of the terms in this sum will not appear.

Proof. If $j - 1 \leqslant k \leqslant j + 1$ then

$$\begin{aligned} |x - x_k| S_k(x) &\leqslant |x - x_k| = |\cos\theta - \cos\theta_k| \\ &\leqslant |\theta - \theta_k| = O(|\theta_j - \theta_k|) = O(1/n). \end{aligned}$$

The lemma now follows immediately.

LEMMA 2.

$$\sum_{k=1}^{j-2} |x - x_k| F_k(x) = O(1/n)$$

and

$$\sum_{k=j-2}^{n} |x - x_k| F_k(x) = O(1/n).$$

Proof. Using Eq. (8) in [5], we find, for $1 \leqslant k \leqslant j - 2$,

$$|x - x_k| F_k(x) = O\left(\frac{j-k}{n}\right) \cdot O((j-k)^{-4}),$$

and hence

$$\sum_{k=1}^{j-2} |x - x_k| F_k(x) = O(1/n) \sum_{i=1}^{\infty} i^{-3} = O(1/n)$$

The second part of the lemma may be proved in a similar manner.

LEMMA 3.

$$\sum_{k=1}^{j-2} |x - x_k| H_k(x) = O(1/n)$$

and

$$\sum_{k=j-2}^{n} |x - x_k| H_k(x) = O(1/n).$$

Proof. By using Eq. (10) in [5], one can prove Lemma 3 in the same way that Lemma 2 was established.

LEMMA 4. *If* $-1 \leqslant x = \cos\theta \leqslant 1$, *then*

$$\sum_{k=1}^{j-2} |x - x_k| \, G_k(x) = \frac{4n^2-1}{6\pi n^3} \cos^4 n\theta \cdot \sin\theta \cdot \ln j + O(1/n) \tag{4.1}$$

and

$$\sum_{k=j+2}^{n} |x - x_k| \, G_k(x) = \frac{4n^2-1}{6\pi n^3} \cos^4 n\theta \cdot \sin\theta \cdot \ln(n-j) + O(1/n). \tag{4.2}$$

Proof. Note that if $j = 1, 2, n-1$, or n, then one of these sums will not be required. We will prove (4.1) only, since the proof of (4.2) is very similar. For $1 \leqslant k \leqslant j-2$, $x_k > x$, and therefore

$$\begin{aligned}
&|x - x_k| \, G_k(x) \\
&= \frac{4n^2-1}{6n^4} \cdot T_n(x)^4 \cdot \frac{1 - x x_k}{x_k - x} \\
&= \frac{4n^2-1}{6n^4} \cdot T_n(x)^4 \cdot \frac{1 - x^2}{x_k - x} + \frac{4n^2-1}{6n^4} \cdot T_n(x)^4 \cdot \frac{x(x - x_k)}{x_k - x} \\
&= M_k(x) - N_k(x),
\end{aligned} \tag{4.3}$$

say, and

$$N_k(x) = O(n^{-2}).$$

If we let $x = \cos\theta$ $(0 \leqslant \theta \leqslant \pi)$ and $x_k = \cos\theta_k$, then

$$M_k(x) = \frac{4n^2-1}{6n^4} \cos^4 n\theta \, \frac{\sin^2\theta}{\cos\theta_k - \cos\theta}. \tag{4.4}$$

Therefore, by approximating the integral below by an appropriate Riemann sum, we obtain

$$\begin{aligned}
\sum_{k=1}^{j-2} M_k(x) &= \frac{4n^2-1}{6\pi n^3} \cdot \cos^4 n\theta \cdot \sin^2\theta \\
&\quad \cdot \left[\int_0^\theta \frac{1}{\cos y - \cos\theta} - \frac{1}{(\theta - y)\sin\theta} \, dy + O(1) \right] \\
&\quad + \frac{4n^2-1}{6n^4} \cos^4 n\theta \cdot \sin\theta \cdot \sum_{k=1}^{j-2} (\theta - \theta_k)^{-1} \\
&= \frac{4n^2-1}{6n^4} \cos^4 n\theta \cdot \sin^2\theta \cdot \sum_{k=1}^{j-2} (\theta - \theta_k)^{-1} + O(1/n).
\end{aligned} \tag{4.5}$$

If $\theta = \pi$, then (4.1) follows from (4.5). For $\theta < \pi$ we have, for n sufficiently large,

$$\sum_{k=1}^{j-2} (\theta_{j-1} - \theta_k)^{-1} < \sum_{k=1}^{j-2} (\theta - \theta_k)^{-1} < \sum_{k=1}^{j-2} (\theta_{j-1} - \theta_k)^{-1}.$$

Therefore,

$$\sum_{k=3}^{j} 1/k < \frac{\pi}{n} \sum_{k=1}^{j-2} (\theta - \theta_k)^{-1} < \sum_{k=1}^{j-2} 1/k.$$

and hence

$$\frac{\pi}{n} \sum_{k=1}^{j-2} (\theta - \theta_k)^{-1} = \ln j + O(1). \tag{4.6}$$

From (4.5) and (4.6) it follows that

$$\sum_{k=1}^{j-2} M_k(x) = \frac{4n^2 - 1}{6\pi n^3} \cos^4 n\theta \cdot \sin\theta \cdot \ln j - O(1/n). \tag{4.7}$$

Estimate (4.1) now follows from (4.3), (4.4), and (4.7).

LEMMA 5. *If* $-1 \leqslant x = \cos\theta \leqslant +1$, *then*

$$\begin{aligned} W_n(x) &= \left(\sum_{k=1}^{j-2} - \sum_{k=j+2}^{n}\right)(|x - x_k| G_k(x)) \\ &= \frac{4}{3\pi} \cos^4 n\theta \cdot \sin\theta \cdot \frac{\ln n}{n} + O(1/n). \end{aligned}$$

Proof. From Lemma 4 we have

$$W_n(x) = \frac{4n^2 - 1}{6\pi n^3} \cos^4 n\theta \cdot \sin\theta \cdot \ln(j(n-j)) + O(1/n). \tag{4.8}$$

But

$$j(n-j) = n^2 \cdot \frac{j}{n} \cdot \frac{n-j}{n} = n^2 \left(\frac{\theta}{\pi} + O(1/n)\right)\left(\frac{\pi - \theta}{\pi} + O(1/n)\right),$$

and hence

$$\ln(j(n-j)) = 2\ln n + \ln\theta + \ln(\pi - \theta) + O(1). \tag{4.9}$$

We now observe that

$$\frac{4n^2 - 1}{6\pi n^3} = \frac{2}{3\pi n} + O(n^{-3}), \tag{4.10}$$

$$\sin\theta \cdot \ln\theta = O(1), \qquad 0 \leqslant \theta \leqslant \pi, \tag{4.11}$$

and

$$\sin\theta\cdot\ln(\pi-\theta)=O(1),\qquad 0\leqslant\theta\leqslant\pi. \tag{4.12}$$

The lemma now follows from (4.8) through (4.12).

Theorem 3 finally follows from (3.7) and Lemmas 1, 2, 3, and 5. Corollary 2 mentioned in Section 2 follows from Theorem 3 once we notice that

$$\sup\{\|R_{4n-1}(f)-f\| : f\in \text{Lip } 1\} = \max\{\Delta_n(x): -1\leqslant x\leqslant +1\}.$$

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